A QED Shower Including the Next-to-leading Logarithm Correction in e^+e^- Annihilation

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ABSTRACT

We develop an event generator, NLL-QEDPS, based on the QED shower including the next-to-leading logarithm correction in the e^+e^- annihilation. The shower model is the Monte Carlo technique to solve the renormalization group equation so that they can calculate contributions of $\alpha^m \log^n(S/m_e^2)$ for any m and n systematically. Here α is the QED coupling, m_e is the mass of electron and S is the square of the total energy in the e^+e^- system. While the previous QEDPS is limited to the leading logarithm approximation which includes only contributions of $(\alpha \log(S/m_e^2))^n$, the model developed here contains terms of $\alpha(\alpha \log(S/m_e^2))^n$, the the next-to-leading logarithm correction. The shower model is formulated for the initial radiation in the e^+e^- annihilation. The generator based on it gives us events with q^2, which is a virtual mass squared of the virtual photon and/or Z-boson, in accuracy of 0.04%, except for small q^2/S.

1 Introduction

In high energy reactions with electron beams, it is important to study radiative corrections[1]. For this study event generators are indispensable tools. We have made the event generator, QEDPS [2]-[5], for radiative corrections in the e^+e^- annihilation based on the shower model, which can radiate any number of photons. However, this model is limited to the leading logarithm(LL) approximation. In this paper we develop a shower model in the next-to-leading logarithm(NLL) approximation[7]-[10]. The magnitude of the NLL order correction is of $\alpha^2/\pi^2 \log(S/m_e^2)$, which is about 0.0001, if $\sqrt{S}$ is 100GeV. So the NLL shower might be irrelevant to actual measurements. However this is wrong since contributions due to soft photons are large. They are estimated to be $\alpha^2/\pi^2 \log(S/m_e^2)\log^2(E_\gamma/\sqrt{S})$, which is about 0.005 if the measured energy E_γ for a observed photon is 100MeV. This value is not negligible in precise experiments.

In this paper we limit the event generator in the NLL approximation to the e^+e^- annihilation, especially to the radiative process on the initial state. Applications of the NLL shower to other process such as the Bhabha scattering [3] are discussed in other papers.

Our study is completely based on the renormalization group equation(RGE), which has been developed well in QCD[6]. First we clarify meanings of the NLL order approximation. Let us consider a dimensionless observable $F(Q^2/\mu^2, \alpha_0)$ with the mass scale Q^2 and the renormalization point μ^2. If the coupling α_0 is small at μ^2 and the ratio Q^2/μ^2 is large, the RGE shows us that $F(Q^2/\mu^2, \alpha_0)$ can be expanded by the coupling constant after summing terms of $[\alpha_0 \log(Q^2/\mu^2)]^n$ for all n.

$$\begin{aligned} F(Q^2/\mu^2, \alpha_0) &= F^{(1)}(\alpha_0 \log(Q^2/\mu^2)) + \alpha_0 F^{(2)}(\alpha_0 \log(Q^2/\mu^2)) \\ &+ \alpha_0^2 F^{(3)}(\alpha_0 \log(Q^2/\mu^2)) + ... \end{aligned} \quad (1)$$

The first term is the LL order approximation. If the the second term is included, the approximation is of the NLL order.

Sect.2 contains discussions on the RGE and the formulas needed for later sections. In Sect.3 we apply them to the e^+e^- annihilation process and give quantities such as the anomalous dimensions explicitly. In Sect.4 we formulate the shower model

briefly. In Sect.5 we discuss the singular behavior of the NLL order correction. The simple perturbative expansion breaks down due to effects of soft photons so that more sophisticated techniques are introduced. Then the effective shower model in the LL order is given. At this stage the model has the same form as that in the LL order model. But some constraints are imposed so that it has included some contributions of the NLL correction. Sect.6 contains results of the effective shower model. In Sect.7 we present the explicit form of the second order P function used in the NLL shower model. In Sect.8 we construct the event generator by defining kinematical variables in terms of the variable in the shower model. Also some problems on the construction are pointed out. The NLL approximation needs the second order coefficient in the $\beta-$function of the coupling, but we drops contributions from this coefficient, which is discussed in Sect.9. In Sect.10 we present a method to compensate results by the shower because they contain the Q^2-independent contribution due to the constraint. Sect.11 is devoted to conclusions and discussions, where numerical results in our study are summarized as well as limitations of our model are given. Also we make some comments on applications of our model to QCD.

We present three appendices for some technical parts of our model. Appendix A gives us relations between the usual perturbative expansion and the RGE. In Appendix B we discuss the approximation that is made in order to get analytical expressions for results by the shower model. In Appendix C we present compact descriptions for the shower algorithm, which we apply to compensating results for the Q^2-independent contribution.

2 Renormalization group equation

In the RGE[6], the value of the coupling depends on μ^2, so that it is not constant, but the function of μ^2, $\alpha(\mu^2)$. Then the derivative of the coupling by μ^2 is given by the function of the coupling only.

$$\mu^2 \frac{d\alpha(\mu^2)}{d\mu^2} = \beta(\alpha(\mu^2)).$$

Solving this equation, we obtain that

$$\log(Q^2/\mu^2) = \int_{\alpha_0}^{\bar{\alpha}} \frac{1}{\beta(\alpha)} d\alpha, \tag{2}$$
$$\bar{\alpha} = \alpha(Q^2), \quad \alpha_0 = \alpha(\mu^2).$$

Here $\bar{\alpha}$ is called the running coupling. If one applies the RGE to the dimensionless observable $F(Q^2/\mu^2, \alpha_0)$, the following equation is obtained.

$$(\mu^2\partial/\partial\mu^2 + \beta(\alpha_0)\partial/\partial\alpha_0 - \gamma(\alpha_0))F(Q^2/\mu^2, \alpha_0) = 0. \tag{3}$$

$\gamma(\alpha)$ is the anomalous dimension which is the function of the coupling α only and depends on a process. Then we solve this equation to obtain that

$$F(Q^2/\mu^2, \alpha_0) \ = \ \exp(-\int_{\alpha_0}^{\bar{\alpha}} \gamma(\alpha)/\beta(\alpha) d\alpha) F(1, \bar{\alpha}). \tag{4}$$

In the NLL order,

$$\beta(\alpha) = \beta_1\alpha^2 + \beta_2\alpha^3,$$
$$\beta_1 = \frac{1}{3\pi}, \quad \beta_2 = \frac{1}{2\pi^2},$$
$$\gamma(\alpha) = \gamma_1\alpha + \gamma_2\alpha^2.$$

Solving Eq.(2) on the running coupling,

$$\beta_1 \log(Q^2/\mu^2) = -(\frac{1}{\bar{\alpha}} - \frac{1}{\alpha_0}) - \frac{\beta_2}{\beta_1}\log(\bar{\alpha}/\alpha_0). \tag{5}$$

If α_0 is small but $\alpha_0 \log(Q^2/\mu^2)$ is large, we keep any term of $[\alpha_0 \log(Q^2/\mu^2)]^n$ and drop terms of $\alpha_0^K \log(Q^2/\mu^2)(K \geq 3)$. Then we obtain the explicit formula for the coupling at Q^2 in the NLL order.

$$\bar{\alpha} = \frac{\alpha_0}{1 - \alpha_0\beta_1 \log(Q^2/\mu^2)}\{1 - \frac{\alpha_0\beta_2}{\beta_1}\frac{\log[1 - \alpha_0\beta_1\log(Q^2/\mu^2)]}{1 - \alpha_0\beta_1\log(Q^2/\mu^2)}\}. \tag{6}$$

The integral inside the exponential of (4) on the anomalous dimension is carried out in the NLL order

$$I_\gamma = \int_{\alpha_0}^{\bar{\alpha}} \gamma(\alpha)/\beta(\alpha) d\alpha = \frac{\gamma_1}{\beta_1}\log(\bar{\alpha}/\alpha_0) + (\frac{\gamma_2}{\beta_1} - \frac{\gamma_1\beta_2}{\beta_1^2})(\bar{\alpha} - \alpha_0). \tag{7}$$

The Q^2 dependence of $F(Q^2/\mu^2, \alpha_0)$ can be expressed by $\beta_1, \beta_2, \gamma_1$, γ_2 and notations used in Appendix A.

$$\begin{aligned} F(Q^2/\mu^2, \alpha_0) \ &= \ \exp[-\int_{\alpha_0}^{\overline{\alpha}} d\alpha \frac{\gamma(\alpha)}{\beta(\alpha)}] F(1, \overline{\alpha}) \\ &= \ \exp[-\frac{\gamma_1}{\beta_1}\log(\frac{\overline{\alpha}}{\alpha_0}) - (\frac{\gamma_2}{\beta_1} - \frac{\gamma_1\beta_2}{\beta_1^2})(\overline{\alpha} - \alpha_0)](f_0 + f_1^0\overline{\alpha}). \end{aligned} \tag{8}$$

To obtain the formula for the Q^2 dependence, we take the ratio of $F(Q^2/\mu^2, \alpha_0)$ and $F(1, \alpha_0)$.

$$\begin{aligned} & F(Q^2/\mu^2, \alpha_0)/F(1, \alpha_0) \\ = \;& \exp[-\frac{\gamma_1}{\beta_1}\log(\frac{\overline{\alpha}}{\alpha_0}) - (\frac{\gamma_2}{\beta_1} - \frac{\gamma_1\beta_2}{\beta_1^2})(\overline{\alpha} - \alpha_0)](f_0 + f_1^0\overline{\alpha})/(f_0 + f_1^0\alpha_0). \end{aligned} \tag{9}$$

Here since $(f_0 + f_1^0\overline{\alpha})/(f_0 + f_1^0\alpha_0)$ can be approximated by $1 + \frac{f_1^0}{f_0}(\overline{\alpha} - \alpha_0)$,

$$\begin{aligned} & F(Q^2/\mu^2, \alpha_0)/F(1, \alpha_0) \\ = \;& \exp[-\frac{\gamma_1}{\beta_1}\log(\frac{\overline{\alpha}}{\alpha_0}) - (\frac{\gamma_2}{\beta_1} - \frac{\gamma_1\beta_2}{\beta_1^2})(\overline{\alpha} - \alpha_0)][1 + \frac{f_1^0}{f_0}(\overline{\alpha} - \alpha_0)]. \end{aligned} \tag{10}$$

Then we expand terms of $\overline{\alpha}-\alpha_0$ in the exponent and drop terms of $(\overline{\alpha}-\alpha_0)^K (K \geq 2)$.

$$\begin{aligned} & F(Q^2/\mu^2, \alpha_0)/F(1, \alpha_0) \\ = \;& \exp[-\frac{\gamma_1}{\beta_1}\log(\frac{\overline{\alpha}}{\alpha_0})]\{1 + (-\frac{\gamma_2}{\beta_1} + \frac{\gamma_1\beta_2}{\beta_1^2} + \frac{f_1^0}{f_0})(\overline{\alpha} - \alpha_0)\}. \end{aligned} \tag{11}$$

Finally we describe the explicit expression of the Q^2-dependence for $F(Q^2/\mu^2, \alpha_0)$.

$$\begin{aligned} & F(Q^2/\mu^2, \alpha_0)/F(1, \alpha_0) \\ = \;& \exp[\frac{\gamma_1}{\beta_1}\log(1 - \alpha_0\beta_1\log(Q^2/\mu^2)) + \frac{\gamma_1\beta_2\alpha_0}{\beta_1^2}\frac{\log(1 - \alpha_0\beta_1\log(Q^2/\mu^2))}{1 - \alpha_0\beta_1\log(Q^2/\mu^2)}] \\ \{ \;& 1 + (-\frac{\gamma_2}{\beta_1} + \frac{\gamma_1\beta_2}{\beta_1^2} + \frac{f_1^0}{f_0})\frac{\beta_1\alpha_0^2\log(Q^2/\mu^2)}{1 - \alpha_0\beta_1\log(Q^2/\mu^2)}\}. \end{aligned} \tag{12}$$

The above is the fundamental equation for the NLL shower model.

3 Annihilation

In this section we present explicit formulas for the annihilation cross section. When $\sigma_0(Q^2)$ is the bare cross section, i.e. the cross section without any radiative correction, the observed cross section is expressed by the structure function $D(x, Q^2)$ and the coefficient function $C(x, \alpha)$ [11].

$$\frac{d\sigma_{obs}(S, Q^2)}{dQ^2} = \sigma_0(Q^2)\frac{1}{S}\int_0^1 dx_1\frac{1}{x_1}\int_0^1 dx_2\frac{1}{x_2}D(x_1, Q^2)D(x_2, Q^2)C(z, \bar{\alpha}), \tag{13}$$

$$z = \frac{\tau}{x_1 x_2} = \frac{Q^2}{s}, \qquad \tau = \frac{Q^2}{S}, \quad s = x_1 x_2 S,$$

where S is the total energy squared.

In oder to solve the RGE in the NLL order analytically, we take moments of Eq.(13).

$$\frac{d\sigma_{obs}(n, Q^2)}{dQ^2} = \frac{\sigma_0(Q^2)}{Q^2} D(n, Q^2) D(n, Q^2) C(n, \bar{\alpha}). \tag{14}$$

Here they are given by taking moments.

$$\frac{d\sigma_{obs}(n)}{dQ^2} = \int_0^1 d\tau \frac{d\sigma_{obs}(S, Q^2)}{dQ^2} \tau^n, \tag{15}$$

$$D(n, Q^2) = \int_0^1 \frac{dx}{x} x^n D(x, Q^2), \tag{16}$$

$$C(n, \alpha) = \int_0^1 \frac{dz}{z} z^n C(z, \alpha). \tag{17}$$

$D(n, Q^2)$ corresponds to the exponential term in Eq.(4), while $C(1, \bar{\alpha})$ does to $F(1, \bar{\alpha})$ so that

$$D(n, Q^2) = \exp[-\int_{\alpha_0}^{\bar{\alpha}} \frac{\gamma(n, \alpha)}{\beta(\alpha)} d\alpha], \tag{18}$$

$$\gamma(n, \alpha) = \gamma_1(n)\alpha + \gamma_2(n)\alpha^2, \tag{19}$$

$$C(n, \alpha) = 1 + \frac{\alpha}{2\pi} C_1(n). \tag{20}$$

Then we obtain the Q^2-dependence of the cross section, as in Eq.(11).

$$\begin{aligned} \frac{1}{\sigma_0(Q^2)} Q^2 \frac{d\sigma_{obs}(n, Q^2)}{dQ^2} &= \exp[-2\frac{\gamma_1(n)}{\beta_1} \log(\frac{\overline{\alpha}}{\alpha_0})] \\ \times \quad \{1 + (-2\frac{\gamma_2(n)}{\beta_1} &+ 2\frac{\gamma_1(n)\beta_2}{\beta_1^2} + \frac{1}{2\pi} C_1(n))(\overline{\alpha} - \alpha_0)\}. \end{aligned} \tag{21}$$

Here we summarize $\gamma_1(n), \gamma_2(n)$ and other quantities needed for the annihilation process. In order to get the DGLAP equation, which is used in the shower model, we replace the variable μ^2 by Q^2 in the RGE, so that the equation of $D(n, Q^2)$ on Q^2 is

$$Q^2 \frac{d}{dQ^2} D(n, Q^2) = (Q^2 \frac{\partial}{\partial Q^2} + \beta(\bar{\alpha}) \frac{\partial}{\partial \bar{\alpha}}) D(n, Q^2) = -\gamma(n, \bar{\alpha}) D(n, Q^2). \tag{22}$$

Of course $D(n,Q^2)$ of Eq.(18) satisfies the above equation. The inverse Mellin transformation of the above is called the DGLAP equation.

$$Q^2\frac{dD(x,Q^2)}{dQ^2} = \int_x^1 \frac{dy}{y}P(x/y,\bar{\alpha}))D(y,Q^2). \tag{23}$$

Here $P(x,\alpha)$ is called P function, which is defined by the anomalous dimension $\gamma(n,\alpha)$;

$$\gamma(n,\alpha) = -\int_0^1 dx x^{n-1}P(x,\alpha).$$

$P(x,\alpha)$ is divided into the LL and the NLL terms.

$$P(x,\alpha) = \frac{\alpha}{2\pi}P^{(1)}(x) + (\frac{\alpha}{2\pi})^2 P^{(2)}(x), \tag{24}$$

$$P^{(1)}(x) = P_+(x),$$

$$P(x) = \frac{1+x^2}{1-x}.$$

Here we introduce the +notation for a function $f(x)$.

$$f_+(x) = f(x) - \delta(1-x)\int_0^1 f(y)dy.$$

The explicit form of $\gamma_1(n)$ is useful for examining the model.

$$\gamma_1(n) = \frac{-1}{2\pi}\int_0^1 dx x^{n-1}P^{(1)}(x) = \frac{1}{2\pi}(2S_1(n-1) - \frac{3}{2} + \frac{1}{n} + \frac{1}{n+1}).$$

Here we introduce functions on the summation.

$$S_m(n) = \sum_{k=1}^{n}\frac{1}{k^m} \quad \text{for} \quad n \geq 1,$$

$$S_m(n) = 0 \quad \text{for} \quad n \leq 0.$$

The NLL term in the moment expression is not so compact so that we present only $P^{(2)}(x)$, which is found in Refs.[7],[10][1].

$$\gamma_2(n) = \frac{-1}{(2\pi)^2}\int_0^1 dx x^{n-1}P^{(2)}(x),$$

$$P^{(2)}(x) = P_{a+}(x) + P_{b+}(x),$$

$$\begin{aligned} P(x)_a &= -P(x)\log(x)\log(1-x) - (\frac{3}{1-x} + 2x)\log(x) \\ &- \frac{1}{2}(1+x)\log^2(x) - 5(1-x), \end{aligned} \tag{25}$$

[1] We drop terms of branching of e^- into e^+, which are quite small.

$$P_b(x) = \frac{2}{3}[P(x)(-\log(x) - \frac{5}{3}) - 2(1-x)], \tag{26}$$

$$C(x,\alpha) = \delta(1-x) + \frac{\alpha}{2\pi}C_1(x).$$

Here $C_1(x)$ has been calculated for the Drell-Yan process in QCD[11], which is the sum of C_{ep+} in the deep-inelastic scattering and an additional term $\Delta C(x)$.

$$C_1(x) = 2C_{ep+}(x) + \Delta C(x),$$

$$C_{ep}(x) = P(x)[\log(\frac{1-x}{x}) - \frac{3}{4}] + \frac{9+5x}{4}, \tag{27}$$

$$\Delta(x) = 2P(x)\log(1-x) - \frac{3}{1-x} - 6 - 4x, \tag{28}$$

$$\Delta C(x) = \Delta_+(x) + \delta(1-x)(-\frac{7}{2} + \frac{4\pi^2}{3}). \tag{29}$$

In our model we make use of the scheme dependence, which says that only the combination of $-2\frac{\gamma_2(n)}{\beta_1} + \frac{1}{2\pi}C_1(n)$ can be predicted by the RGE, but each quantity is not. By making use of this freedom, we can put $\tilde{C}_1(n) = 0$.

$$-2\frac{\gamma_2(n)}{\beta_1} + \frac{1}{2\pi}C_1(n) = -2\frac{\tilde{\gamma}_2(n)}{\beta_1} + \frac{1}{2\pi}\tilde{C}_1(n) = -2\frac{\tilde{\gamma}_2(n)}{\beta_1}. \tag{30}$$

That is

$$\tilde{\gamma}_2(n) = \gamma_2(n) - \frac{\beta_1}{4\pi}C_1(n). \tag{31}$$

Or

$$\tilde{P}^{(2)}(x) = P^{(2)}(x) + \pi\beta_1 C_1(x). \tag{32}$$

In our model we use $\tilde{P}^{(2)}(x)$, which means that the hard cross section is not employed.

4 Shower model

The shower model that we call in this paper is the Monte Carlo method to solve the DGLAP equation by repeating the branching that the electron branches out into the child electron and the photon, where a variable x and a virtual mass squared K^2 are generated. Here x is the energy fraction of the child to one of the parent, while K^2 is the absolute value of the virtual mass squared of the child. So the moment distribution of $x_b^{n-1} = (x_1 x_2 \cdots x_L)^{n-1}$ calculated by the shower model agrees with the analytic result of the RGE completely within the statistical error. Here x_i is x at the $i-$th branching and L is a number of branchings in one branching process.

The shower model needs the scheme to cutoff the infrared singularity, though it is arbitrary. Since we would like to apply the shower model to the event generator that produces electrons, photons and other particles in simulations, we adopt the following cutoff scheme.

$$x < 1 - \mu^2/K^2.$$

The definition of x, which is necessary to construct the generator, is given in Sect.8.

Also our shower model employs the double cascade scheme, in order that the electron and the positron make the branching process independently[8],[2],[4]. In this scheme we impose the constraint to x.

$$1 - x > K^2/Q^2.$$

By using these constraints, we can apply the Monte Carlo method to generate x and K^2, as described in Ref.[2].

5 The singular behavior of the NLL order correction

In Sect.3 we presented the P function in Eq.(32), which we use in the shower model. As $x \to 1$, the most singular behavior of this function is

$$(\frac{\alpha(K^2)}{2\pi})^2 \; 8\pi\beta_1 \; \frac{\log(1-x)}{1-x}.$$

This is dangerous, since it can become larger than the singular LL term, which is

$$\frac{\alpha(K^2)}{2\pi}\frac{2}{1-x}.$$

In order to make the branching stable, it was suggested to use the running coupling $\alpha((1-x)K^2)$ instead of $\alpha(K^2)$ in order to include term of $\alpha^n \frac{\log^n(1-x)}{1-x}$ into the model in Ref.[7]. By this replacement, the singular behavior of P function is

$$\begin{aligned}
&\frac{\alpha(K^2)}{2\pi}\frac{2}{1-x}+A(\frac{\alpha(K^2)}{2\pi})^2\frac{\log(1-x)}{1-x} \\
&= \frac{\alpha_0}{2\pi[1-\beta_1\alpha_0\log(K^2/\mu^2)]}\frac{2}{1-x}+A(\frac{\alpha(K^2)}{2\pi})^2\frac{\log(1-x)}{1-x} \\
&= \frac{\alpha_0}{2\pi[1-\beta_1\alpha_0\log((1-x)K^2/\mu^2)]}\frac{2}{1-x}+(A-4\pi\beta_1)(\frac{\alpha(K^2)}{2\pi})^2\frac{\log(1-x)}{1-x} \\
&\quad +O(\alpha_0^3).
\end{aligned}$$

In the deep inelastic scattering, indeed $A = 4\pi\beta_1$, so that the dangerous term disappears after the replacement. On the other hand we have $A = 8\pi\beta_1$ in the annihilation. By this method $4\pi\beta_1$ of them can be included into the running coupling. We have to include the remnant $4\pi\beta_1$ terms into the effective LL form in order to remove the dangerous term in the second order P function. This can be done by taking account of the kinematical constraint in the annihilation as follows.

In the annihilation, both electron and positron to radiate photons.

$$e^-(P_1)+e^+(P_2)\to e^-(p_1)+e^+(p_2)+X\to\gamma(q)+X.$$

This implies that the spacelike virtual electron(p_1) and positron (p_2) annihilate into the virtual photon(q). Although we can calculate q^2 by p_1, p_2, as seen in Fig.1, we make the approximation that

$$q^2\approx x_{b1}x_{b2}(1-t_1)(1-t_2)S. \tag{33}$$

Here $t_1=-p_1^2/Q^2(t_2=-p_2^2/Q^2)$. The accuracy of this approximation is discussed in Sect.8.

Therefore our shower model calculates moments with respect to the variable $x_b(1-t)$. The $n-$th moment $(x_b(1-t))^{n-1}$ should agree with the structure function

$D_s(n,Q^2)$ on the electron(See ref.[4]).

$$
\begin{aligned}
& D_s(n,Q^2) = \Pi(Q^2,\mu^2) + \int_0^1 \frac{dt}{t}(1-t)^{n-1}\Pi(Q^2,tQ^2) \\
& \qquad\quad \times \int_0^1 dx \frac{\alpha((1-x)tQ^2)}{2\pi} P(x)x^{n-1}\theta(1-x-t)\theta((1-x)t-\epsilon) \\
& \times \exp\{\int_0^t \frac{dt'}{t'}\int dx' \frac{\alpha((1-x')t'Q^2)}{2\pi}P(x')(x'^{n-1}-1) \\
& \times\theta(1-x'-t')\theta((1-x')t'-\epsilon)\},
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
\Pi(Q^2,Q_0^2) &= \exp[-\int_{Q_0^2/Q^2}^1 \frac{dt}{t}\int dx \frac{\alpha((1-x)tQ^2)}{2\pi} \\
& \quad \times P(x)\theta(1-x-t)\theta((1-x)t-\epsilon)].
\end{aligned} \tag{35}
$$

Here $\epsilon = \mu^2/Q^2$. $\Pi(Q^2,Q_0^2)$ is the non-branching probability that the electron does not branch for possible virtual mass squared between Q^2 and Q_0^2. In Eq.(34) the first term represents the no-branching case so that the moment is unity for any n. The front term on the exponential does the last branching, while the exponential appears after repeating branchings. The reason for the special form on the last branching is that there the virtual mass squared is involved in the moment, as seen in Eq.(33).

In order to obtain the expression that is possible to be calculated analytically, we approximate $(1-t)^n$ by $\theta(1/n-t)$.

$$
\begin{aligned}
D_s(n,Q^2) &= \exp\{\int_0^1 \frac{dt}{t}\int_0^1 dx \frac{\alpha((1-x)tQ^2)}{2\pi} \\
& \quad \times P(x)[(x(1-t))^{n-1}-1]\theta(1-x-t)\theta((1-x)t-\epsilon)\}.
\end{aligned} \tag{36}
$$

The error due to this approximation is discussed in Appendix B. We write $D_s(n,Q^2) = \exp[I_s(n,Q^2)]$, and we perform the integrals for $I_s(n,Q^2)$.

$$
\begin{aligned}
& I_s(n,Q^2) = \\
= & \int_0^1 dz(z^{n-1}-1)\int_0^1 \frac{dt}{t}\int_0^1 dx\delta(z-x(1-t))\frac{\alpha((1-x)tQ^2)}{2\pi}P(x) \\
\times & [(x(1-t))^{n-1}-1]\theta(1-x-t)\theta((1-x)t-\epsilon).
\end{aligned} \tag{37}
$$

Then we integrate $I_s(n,Q^2)$ over x and t, where we assume that n is not so large so that contributions from regions of $z \sim 1$ can be neglected. Also we neglect $O(\epsilon)$ terms. Finally we obtain

$$
I_s(n,Q^2)
$$

$$= \frac{1}{2\pi}\int_0^1 dx(x^{n-1}-1)P(x)\frac{1}{\beta_1}\log(\frac{\overline{\alpha}}{\alpha_0}) + \frac{\overline{\alpha}}{2\pi}\int_0^1 dx(x^{n-1}-1)\Delta P(x), \quad (38)$$

where

$$\Delta P(x) = 2P(x)\log(1-x) - \frac{1}{1-x}\log x$$
$$+ \quad (1+x)\log(1+\sqrt{x}) + \frac{1}{2}(1+x)\log x - \sqrt{x} + x. \quad (39)$$

In Eq.(38), the first term is the LL order while the the second order is the NLL order, though it contains the Q^2-independent contribution. By noting that $\overline{\alpha} = (\overline{\alpha}-\alpha_0)+\alpha_0$ and

$$\int_{\mu^2/Q^2}^1 \frac{dt}{t}[\alpha(tQ^2)]^2 = \frac{1}{\beta_1}(\overline{\alpha}-\alpha_0), \quad (40)$$

one can see that the second term has $(\alpha/2\pi)^2\log(1-x)/(1-x)$ and its coefficient is $8\pi\beta_1$ as expected.

6 Results by the shower in the effective LL order

Summarizing the discussion in the previous section, Eq.(38) was derived by adopting the three schemes:

1) to use $(1-x)K^2$ for the argument of the running coupling, i.e. at the branching we employ the following coupling,

$$\alpha((1-x)K^2) = \frac{\alpha_0}{1-\beta_1\alpha_0\log((1-x)K^2/\mu^2)}.$$

As pointed out in Ref.[7], $(1-x)K^2$ is about the transverse momentum squared at the branching.

2) the double cascade scheme[8], where we impose the constraint of

$$(1-x) > K^2/Q^2.$$

3) to define q^2 in the annihilation as

$$q^2 = x_{b1}(1-t_1)x_{b2}(1-t_2)S,$$

$$x_b = 1 \times x_1 \times \cdots \times x_L.$$

The moment of $x_b(1-t)$ by this shower in Eq.(38) is calculated in the analytic form.

$$\begin{aligned}
I_s(n,Q^2) &= \frac{1}{2\pi\beta_1}\log(\frac{\overline{\alpha}}{\alpha_0})[-2S_1(n-1)-\frac{1}{n}-\frac{1}{n+1}+\frac{3}{2}] \\
&+ \frac{\overline{\alpha}}{2\pi}[2(S_1^2(n-1)+S_2(n-1)+\frac{1}{n}S_1(n)+\frac{1}{n+1}S_1(n+1)-\frac{7}{4}) \\
&- S_2(n-1)+\frac{1}{n}(-S_1(n)+S_1(2n))+\frac{1}{n+1}(-S_1(n+1)+S_1(2n+2)) \\
&- \frac{1}{2}(\frac{1}{n^2}+\frac{1}{(n+1)^2})-\frac{2}{2n+1}+\frac{1}{n+1}].
\end{aligned} \tag{41}$$

Some comparisons between analytic calculations, Eq.(41), and results by the shower model are shown in Table 1. There we assumed that $\mu^2 = 0.25\times 10^{-6}\mathrm{GeV}^2$ and $\alpha_0 = 1/137$. In Monte Carlo simulations a total number of events is 10^8 and errors are estimated by calculating the variance in 10 data sets of 10^7 events. The agreement of order of 10^{-5} justifies our discussion.

The structure function given by Eq.(41) has the Q^2-independent contribution, $D^f(n)$, which does not vanish at $Q^2=\mu^2$ because the term of $\overline{\alpha}$ in Eq.(41) remains then. $D^f(n)$ is obtained by replacing $\overline{\alpha}$ by α_0 in Eq.(38).

$$D^f(n) = \exp\{\frac{\alpha_0}{2\pi}\int_0^1 dx(x^{n-1}-1)\Delta P(x)\}. \tag{42}$$

In order to calculate the absolute value as well as the Q^2-dependence for the structure function by the shower model, we have to compensate it for the contribution due to $D^f(n)$. The method for the compensation is discussed in Sect.10.

7 Effective $P^{(2)}(x)$

In this section we give the second order P function $P^{(2)eff}(x)$, which is used in the shower model. Since our model imposes that the coefficient function is zero

so that the NLL contribution can be given by $\tilde{P}^{(2)}(x)$ in Eq.(32). As discussed in Sect.5 and 6, our shower model contains the NLL contribution, $\Delta P(x)$, through $\alpha((1-x)K^2)$, the double cascade scheme and the definition of q^2, as described in Eq.(38). In the NLL shower we employ these schemes so that the structure function $D(n,Q^2) = exp[I(n,Q^2)]$ is given by

$$\begin{aligned}
&I(n,Q^2) = \int_0^1 dx \int_0^1 \frac{dt}{t}\theta(1-x-t)\theta((1-x)t-\epsilon)[(1-t)^{n-1}x^{n-1}-1] \\
&\{\frac{\alpha(t(1-x)Q^2)}{2\pi}P(x) + \frac{\alpha(t(1-x)Q^2)^2}{(2\pi)^2}P^{(2)eff}(x)\}. \qquad (43)
\end{aligned}$$

By performing the integral similar to one in the previous section, we have

$$\begin{aligned}
&I(n,Q^2) = \frac{1}{2\pi\beta_1}\log(\frac{\overline{\alpha}}{\alpha_0})\int_0^1 dx(x^{n-1}-1)P(x) \\
&+\frac{\overline{\alpha}}{2\pi}\int_0^1 (x^{n-1}-1)\Delta P(x) + \frac{1}{(2\pi)^2\beta_1}(\overline{\alpha}-\alpha_0)\int_0^1 dx(x^{n-1}-1)P^{(2)eff}(x) \\
= \quad &\frac{1}{2\pi\beta_1}\log(\frac{\overline{\alpha}}{\alpha_0})\int_0^1 dx(x^{n-1}-1)P(x) \\
&+\frac{1}{(2\pi)^2\beta_1}(\overline{\alpha}-\alpha_0)\int_0^1 dx(x^{n-1}-1)[P^{(2)eff}(x)+2\pi\beta_1\Delta P(x)] \\
&+\frac{\alpha_0}{2\pi}\int_0^1 dx(x^{n-1}-1)\Delta P(x). \qquad (44)
\end{aligned}$$

Here we neglected terms of order of α_0^2 or $\overline{\alpha}^2$. $\tilde{P}^{(2)}(x)$ in Eq.(32) should equal to $P^{(2)eff}(x) + 2\pi\beta_1\Delta P(x)$. Therefore $P^{(2)eff}(x)$ is given by

$$\begin{aligned}
&P^{(2)eff}(x) = \tilde{P}^{(2)}(x) - 2\pi\beta_1\Delta P(x) \\
= \quad &P_a(x) + P_b(x) \\
+ \quad &2\pi\beta_1[-\frac{3}{1-x} - P(x)\log(x) + \frac{\log(x)}{1-x} - (1+x)\log(x+\sqrt{x}) + \sqrt{x} - x], \qquad (45)
\end{aligned}$$

which is free from the singular term $\log(1-x)/(1-x)$, as is expected.

8 Event generator

In this section we present the event generator based on the shower model, which was described in Sect.4-7. In order to determine four momenta of the produced

particles we define x to be a fraction of $+(-)$ component of lightcone variables for electrons(positron).

$$x = \frac{p_+}{P_+}(\frac{p_-}{P_-}), \tag{46}$$

$$p_\pm = \frac{E \pm p_z}{\sqrt{2}}. \tag{47}$$

Here four momenta are denoted as (p_x, p_y, p_z, E) and $P_-(P_+)$ denotes a lightcone variable of the initial electron(positron) [2]. At the branching of $e^-(y, -K^2) \rightarrow e^-(xy, -K'^2) + \gamma(y(1-x), 0)$, the momentum conservation imposes the following equation.

$$-K^2 = \frac{-K'^2}{x} + \frac{\vec{k_T}^2}{x(1-x)}, \tag{48}$$

where $\vec{k_T}$ is the transverse momentum, (p_x, p_y). Here note that the electron during the branching process is spacelike. Our cutoff scheme, $x < 1 - \mu^2/K'^2$, equals to $\vec{k_T}^2 \geq \mu^2$, if $K^2 \ll K'^2$.

Using an arbitrary azimuthal angle ϕ, K^2, K'^2 and x one can determine four momenta of the electron and the photon after the branching.

$$p_{e'\mu} = (xP_+, \frac{-K'^2 + \vec{k}_T^2}{x}, \vec{k_T}), \tag{49}$$

$$p_{\gamma\mu} = ((1-x)P_+, \frac{\vec{k}_T^2}{1-x}, -\vec{k_T}), \tag{50}$$

$$k_x = k_T \cos\phi, \quad k_y = k_T \sin\phi, \tag{51}$$

$$k_T^2 = (1-x)(-xK^2 + K'^2). \tag{52}$$

These equations determine four momenta of all particles completely. This implies that in the annihilation process the four momentum q of the virtual photon and/or Z-boson is the sum of momenta p_1 and p_2 of the electron and positron after the branching process, that is $q = p_1 + p_2$(See Fig.1). Then the virtual mass squared of the four momentum is

$$\begin{aligned} q^2 &= p_1^2 + p_2^2 + 2p_1p_2 \\ &= -K_1^2 - K_2^2 + 2(p_{1+}p_{2-} + p_{1-}p_{2+} - \vec{p}_{1T}\vec{p}_{2T}) \\ &= -K_1^2 - K_2^2 + 2(x_{b1}P_{1+}x_{b2}P_{2-} + \frac{-K_1^2 + \vec{p}_{1T}^2}{2x_{b1}P_{1+}} \frac{-K_2^2 + \vec{p}_{2T}^2}{2x_{b2}P_{2-}} - 2\vec{p}_{1T}\vec{p}_{2T}) \end{aligned}$$

$$= \quad -K_1^2 - K_2^2 + x_{b1}x_{b2}S + \frac{(-K_1^2+\vec{p}_{1T}^2)(-K_2^2+\vec{p}_{2T}^2)}{x_{b1}x_{b2}S} - 2\vec{p}_{1T}\vec{p}_{2T}. \tag{53}$$

The variable $\tau = q^2/S$ equals

$$\tau \quad = \quad -t_1 - t_2 + x_{b1}x_{b2} + \frac{(K_1^2+\vec{p}_{1T}^2)(K_2^2+\vec{p}_{2T}^2)}{x_{b1}x_{b2}S^2} - 2\vec{p}_{1T}\vec{p}_{2T}. \tag{54}$$

Here we used $t = K^2/S$. In the generator the ratio τ is given by the above equation, but not by $x_{b1}x_{b2}$. Although the RGE predicts moments on τ, as described in Sect.3, τ of (54) is not a good variable for the shower model, because it gives us moments on $\tau' = x_{b_1}(1-t_1)x_{b2}(1-t_2)$, as discussed in Sect.5. We present the detailed discussion on the accuracy of the generator, which uses τ.

Let us discuss differences between moments on τ and τ'. First note that the last term, $2\vec{p}_{1T}\vec{p}_{2T}$, in the above equation is zero if averages are took, because angles between these vectors are arbitrary. Next τ can be negative while τ' is always positive. Of course the negative τ is unphysical so that the event with the negative τ is abandoned. We introduce a variable $\overline{\tau}$ that can be negative and whose moments are possible to be calculated analytically, which is

$$\overline{\tau} = (x_{b1} - t_1)(x_{b2} - t_2). \tag{55}$$

In this definition $\overline{\tau}$ might be negative. A case of being negative is counted as an event, but these negative values are replaced by zero. Also the generator can fail to make four momenta for the virtual photon and/or Z-boson, i.e. τ is negative. The failed case is counted as an event and τ is set to be zero. Results on moments of τ, $\overline{\tau}$ and τ' for a total number of 10^8 are presented in Table 2. Differences on data for τ in the generator and $\overline{\tau}$ in the shower are quite small and less than 10^{-5}. So we can conclude that moments of τ in the generator is accounted by those of $\overline{\tau}$. We can estimate analytically differences of moments on $\overline{\tau}$ and τ', which are of order of 10^{-3} and decrease rapidly as n rises. They are

$$\begin{aligned}
&D_d(n,Q^2) = \exp\{2\frac{1}{2\pi}\int_0^1 \frac{dt}{t}\int_0^1 dx\theta(t(1-x)-\epsilon)\theta(1-x-t) \\
&\alpha(t(1-x)Q^2)P(x)[(1-t)^{n-1}x^{n-1}-1]\} \\
&-\exp\{2\frac{1}{2\pi}\int_0^1 \frac{dt}{t}\int_0^1 dx\theta(t(1-x)-\epsilon)\theta(1-x-t) \\
&\alpha(t(1-x)Q^2)P(x)[\theta(x-t)(x-t)^{n-1}-1]\}.
\end{aligned} \tag{56}$$

Here for simplicity we neglect $P^{(2)eff}(x)$. If we neglect the running effect on α and approximate $D_d(n, Q^2)$ by the difference between the first terms in expansions by α,

$$D_d(n, Q^2) \approx \frac{2\alpha_0}{2\pi} \int_0^1 \frac{dt}{t} \int_0^1 dx P(x) \theta(t(1-x) - \epsilon) \theta((1-x) - t) \\ \times [x^{n-1}(1-t)^{n-1} - \theta(x-t)(x-t)^{n-1}]. \quad (57)$$

Under these approximations, $D_d(n, Q^2)$ is independent of Q^2. The values of the expression (57) presented in Table 3 can account for the difference on the moment of τ' and $\overline{\tau}$. For an example, the moment of $n = 2$ in Table 2 is 0.93212, and 0.93118, while Table 3 shows the difference 0.00093, which agrees with the difference.

9 β-function

In this section we estimate contributions by the second order correction β_2 in the beta function. Using Eq.(6) for the running coupling, $\overline{\alpha} - \alpha_0$ is given by

$$\overline{\alpha} - \alpha_0 = \frac{\beta_1 \alpha_0^2 \log(Q^2 \mu^2)}{1 - \alpha_0 \beta_1 \log(Q^2/\mu^2)}. \quad (58)$$

Inserting the explicit expression of the running coupling, the integral on the anomalous dimension I_γ in Eq.(7) becomes

$$I_\gamma = \frac{\gamma_1}{\beta_1} \log \frac{1}{1 - \alpha_0 \beta_1 \log(Q^2/\mu^2)} \\ - \frac{\alpha_0 \beta_2 \gamma_1}{\beta_1^2} \frac{\log[1 - \alpha_0 \beta_1 \log(Q^2/\mu^2)]}{1 - \alpha_0 \beta_1 \log(Q^2/\mu^2)} + (\frac{\gamma_2}{\beta_1} - \frac{\gamma_1 \beta_2}{\beta_1^2}) \frac{\beta_1 \alpha_0^2 \log(Q^2 \mu^2)}{1 - \alpha_0 \beta_1 \log(Q^2/\mu^2)}. \quad (59)$$

In the QED process $\alpha_0 \beta_1 \log(Q^2/\mu^2)$ is small so that the logarithm of the second term is approximated as follows.

$$\log[1 - \alpha_0 \beta_1 \log(Q^2/\mu^2)] \approx -\alpha_0 \beta_1 \log(Q^2/\mu^2). \quad (60)$$

This leads to cancellation of the terms with β_2.

$$I_\gamma \approx \frac{\gamma_1}{\beta_1} \log \frac{1}{1 - \alpha_0 \beta_1 \log(Q^2/\mu^2)} + \gamma_2 \alpha_0^2 \frac{\log(Q^2 \mu^2)}{1 - \alpha_0 \beta_1 \log(Q^2/\mu^2)}. \quad (61)$$

The leading term in the neglected terms is

$$\frac{\gamma_1\beta_2\alpha_0^3}{2}\frac{\log(Q^2/\mu^2)}{1-\alpha_0\beta_1\log(Q^2/\mu^2)},$$

which is less than 10^{-6} if one uses actual values for $\alpha_0, \beta_1, \beta_2,\ \gamma_1$ and $Q^2/\mu^2 = (100\mathrm{GeV}/0.5\mathrm{MeV})^2 = 2.5\times 10^{11}$. Therefore we can neglect the term with β_2 safely.

10 Q^2-independent contribution

As discussed in Sect.5, the shower model contains the Q^2-independent contribution. In other words, the structure function is not $\delta(1-x)$ at $Q^2 = \mu^2$, but its moment is given by $D^f(n)$ of Eq.(42). In study of the radiative corrections on QED, the absolute value of the cross section, not its Q^2-dependence, has to be calculated. So we would like to compensate the structure function for the the Q^2-independent contribution $D^f(n)$. On the moment we only divide results by $D^f(n)$.

$$D^{cmp}(n,Q^2) = D(n,Q^2)/D^f(n).$$

Here $D^{cmp}(x,Q^2)$ is the structure function after the compensation. However, in the generator we need the inverse Mellin transformation. The product of moments is equivalent to the convolution integral of the function in the transformation so

$$\begin{aligned} D^{cmp}(x,Q^2) &= \int_x^1 \frac{dy}{y} D(y,Q^2) D^{\overline{f}}(x/y), \\ (D^f(n))^{-1} &= \int_0^1 dx x^{n-1} D^{\overline{f}}(x). \end{aligned} \tag{62}$$

In the shower the convolution integral is realized by the procedure that x of the initial electron is fixed according to the probability $D^{\overline{f}}(x)$ and then we make the branching process, which induces the structure function.

For performing this we are confronted with three problems. First the explicit form of $D^{\overline{f}}(x)$ is difficult to calculate. Here we use the shower algorithm to get x, which is described in detail in Appendix C.

Second the function $D^{\overline{f}}(x)$ is not positive necessarily so that it could be used as the probability. Equivalently the splitting function $-\Delta P(x)$ in

$$(D^f(n))^{-1} = \exp[-\frac{\alpha_0}{2\pi}\int_0^1 dx(x^{n-1}-1)\Delta P(x)],$$

is negative at x near zero. Since $-\Delta P(x)$ is concentrated near $x \sim 1$, we must be contented with an approximated $\Delta P^A(x)$, which is modified near $x \sim 0$. The approximated $\Delta P^A(x)$ is given by

$$\Delta P^A(x) = \theta(x - x_c)\Delta P(x). \tag{63}$$

Here x_c is fixed to satisfy the condition,

$$\int_0^{x_c} dy \Delta P(y) = 0.$$

The third problem is that the total energy squared S of e^+e^- system is changed by $x_1^f x_2^f S$, where x_1^f and x_2^f are $x-$fractions of the initial electron and positron after the compensation by $D^{\overline{f}}(x)$. Since errors due to this problem are very small numerically, we neglect this problem.

Next we examine numerical results related to $D^f(n)$. First we compare $(D^f(n))^{-1}$ with

$$D^{f,A}(n) = \exp[-\frac{\alpha_0}{2\pi}\int_0^1 (x^{n-1}-1)\Delta P^A(x)], \tag{64}$$

in Table 4. One finds the agreement between results by the shower algorithm and the moment $(D^f(n))^{-1}$ except those for small n. The differences are less than 0.01%. In order to confirm that these differences are due to the second problem, we calculate the moment $D^{f,A}(n)$ through the numerical integration to show these results in the same table. The agreement between values of the second and the third column supports our discussion strongly.

11 Conclusions and discussions

In this paper we have formulated the shower model including the NLL correction in the e^+e^- annihilation and developed the generator. Results on the Q^2-dependence of moments by the generator are summarized in Tables 5, 6 and 7. Here the moment for $n = 1$ is fixed to be unity because it is the event normalization, so that the analytic value is the ratio between the $n-$th moment and the first moment. Analytic results in the NLL order are 0.03% of the LL order ones for small n, but they increase as n rises and are about 0.2% for $n = 100$. The effect of the NLL order is small, but could not be neglected in precise experiments as at LEP or in future colliders. Results of our shower model agree with the analytic calculations of the NLL oder in accuracy of 0.04%. The agreement in the generator is worse for small n, because events might not satisfy with the kinematics constructed by the shower model. Simply speaking, the constructed value for q^2 becomes negative in these events. Table 7 shows differences between moments by the shower model and the analytic one, and those between moments by the generator and the analytic one in the NLL order. Values in the table indicate the magnitude of the systematic error in the present shower and the generator in the NLL order.

Next we mention some comments on limitations of our model. First the accuracy found in q^2 distributions may not be common to other distributions such as the transverse momentum distributions of radiated photons. One of reasons is that we do not include the cross section for the emission of a photon with a large transverse momentum.

Also we neglect effects by the three-body decay in the shower because there is a less interest on detailed distributions of photons. Further we neglect the mixing $P^{(2)}(x)$, which is a contribution that the electron radiates into the spacelike positron with the pair creation, since its effect is expected to be very small.

Third our shower model is limited to the non-singlet case where there are no contributions by radiations with the spacelike photon. A reason for this limitation is that in experiments one can exclude events with the electron positron pair easily, which correspond to pure singlet radiations, as well as that they are quite small. Finally notice that we use S for the mass scale of the RGE, but not q^2.

In this study we have examined our model in the moment form. But from a

experimental view, analyses in the x-space are desired, which will be discussed in coming papers. Also we have to discuss the accuracy of our generator in detail by applying it to several realistic processes such as the muon pair production or the $Z-$Higgs production.

Finally we would like to stress that our study on the NLL shower is quite important for QCD, where the NLL shower has been developed. Because there has been no study on the Q^2-independent contributions by the shower algorithm in QCD. Also precise discussions have led us to the deeper understanding of the shower models. Therefore our study stimulates the interest on further developments of QCD showers.

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Appendix A

In this appendix we explain results of the RGE in terms of the perturbative expansion. In the perturbative expansion by the coupling α_0 at μ^2, the dimensionless quantity $F(Q^2/\mu^2, \alpha_0)$ is calculated up to some order of α_0.

$$F(Q^2/\mu^2, \alpha_0) = f_0 + f_1(Q^2/\mu^2)\alpha_0 + f_2(Q^2/\mu^2)\alpha_0^2 + \cdots \ . \tag{65}$$

Here we assume that the mass scale is only Q^2. In order to compare results of the RGE with those of the perturbative calculation, the ratio $F(Q^2/\mu^2, \alpha_0)/F(1, \alpha_0)$ should be used, because the RGE can calculate the Q^2-dependence only. The ratio is given by

$$\begin{aligned} F(Q^2/\mu^2, \alpha_0)/F(1, \alpha_0) = 1 + [(f_1(Q^2/\mu^2) - f_1(1))/f_0]\alpha_0 + \\ [(f_2(Q^2/\mu^2) - f_2(1))/f_0 + f_1(1)^2/f_0^2 - f_1(Q^2/\mu^2)f_1(1)/f_0^2]\alpha_0^2 + \cdots \ . \end{aligned} \tag{66}$$

The physical quantity contains only $\log(Q^2/\mu^2)$ and the order of power of the logarithm is less than the order of the coupling constant. Therefore

$$f_1(Q^2/\mu^2) = f_1^0 + f_1^1 \log(Q^2/\mu^2), \tag{67}$$

$$f_2(Q^2/\mu^2) = f_2^0 + f_2^1 \log(Q^2/\mu^2) + f_2^2 \log^2(Q^2/\mu^2). \tag{68}$$

If the ratio $F(Q^2/\mu^2, \alpha_0)/F(1, \alpha_0)$ is expressed in terms of $f_1^0, f_1^1, f_2^0, f_2^1$ and f_2^2,

$$\begin{aligned} & F(Q^2/\mu^2, \alpha_0)/F(1, \alpha_0) = 1 + \alpha_0 \frac{f_1^1}{f_0} \log(Q^2/\mu^2) \\ + \quad & \alpha_0^2 \{ \frac{f_2^1}{f_0} \log(Q^2/\mu^2) + \frac{f_2^2}{f_0} \log^2(Q^2/\mu^2) - \frac{f_1^0 f_1^1}{(f_0)^2} \log(Q^2/\mu^2) \}. \end{aligned} \tag{69}$$

The result by the RGE has been discussed in Sect.3 and is summarized by Eq.(12). If we drop terms of α_0^n $(n \geq 3)$ in this equation,

$$\begin{aligned} & F(Q^2/\mu^2, \alpha_0)/F(1, \alpha_0) \\ \approx \quad & 1 - \alpha_0 \gamma_1 \log(Q^2/\mu^2) + \alpha_0^2 \log^2(Q^2/\mu^2)[\frac{1}{2}(\gamma_1)^2 - \gamma_1 \beta_1] \\ + \quad & \alpha_0^2 \log(Q^2/\mu^2)[-\gamma_2 + \frac{f_1^0 \beta_1}{f_0}]. \end{aligned} \tag{70}$$

Since the perturbative expansion (69) should agree with (70) by the RGE, we obtain relations between γ, β and f.

$$\frac{f_1^1}{f_0} = -\gamma_1,$$

$$\frac{f_2^2}{f_0} = \frac{1}{2}(\gamma_1)^2 - \gamma_1\beta_1,$$
$$\frac{f_2^1}{f_0} - \frac{f_1^0 f_1^1}{(f_0)^2} = -\gamma_2 + \frac{f_1^0\beta_1}{f_0}.$$

If we neglect the running effect i.e. $\beta_1 = 0$, we have the simple relation for f_1^1 and f_2^2.

$$f_2^2 = \frac{(f_1^1)^2}{2f_0}. \tag{71}$$

Appendix B

In this appendix we discuss the analytic expression for the moment of $x_b(1-t)$ in the shower model. We need some approximations in order to get the analytic expression for the moment, which is defined in Eq.(34) in Sect.5. The approximated, but analytic expression is given by the expression (36), from which we obtain Eq.(38). A conclusion in this appendix is that the difference between the expression (34) and (36) is of order of α^2 and of n^0, and does not have the Q^2-dependence except one due to the running effect.

First note that the expression (36) equals to

$$\begin{aligned}
&\Pi(Q^2,\mu^2) + \Pi(Q^2,\mu^2)\int_0^1 dt \frac{d}{dt}\exp\{\int_0^t \frac{dt'}{t'}\int_0^1 dx \frac{\alpha((1-x)t'Q^2)}{2\pi}P(x) \\
&(x(1-t'))^{n-1}\theta(1-x-t')\theta((1-x)t'-\epsilon)\} \\
= \;&\Pi(Q^2,\mu^2) + \Pi(Q^2,\mu^2)\int_0^1 \frac{dt}{t}\int_0^1 dx \frac{\alpha((1-x)tQ^2)}{2\pi}P(x) \\
&(x(1-t))^{n-1}\theta(1-x-t)\theta((1-x)t-\epsilon)\exp\{\int_0^t \frac{dt'}{t'}\int_0^1 dx' \frac{\alpha((1-x')t'Q^2)}{2\pi}P(x') \\
&(x'(1-t'))^{n-1}\theta(1-x'-t')\theta((1-x')t'-\epsilon)\}.
\end{aligned} \tag{72}$$

Since the factorization can apply to the non-branching probability, we have

$$\Pi(Q^2,\mu^2) = \Pi(Q^2,tQ^2)\Pi(tQ^2,\mu^2).$$

Then the expression (36) becomes

$$\Pi(Q^2,\mu^2) + \int_0^1 \frac{dt}{t}\Pi(Q^2,tQ^2)\int_0^1 dx \frac{\alpha((1-x)tQ^2)}{2\pi}P(x)$$

$$(x(1-t))^{n-1}\theta(1-x-t)\theta((1-x)t-\epsilon)\exp\{\int_0^t \frac{dt'}{t'}\int_0^1 dx'\frac{\alpha((1-x')t'Q^2)}{2\pi}P(x')$$
$$[(x'(1-t'))^{n-1}-1]\theta(1-x'-t')\theta((1-x')t'-\epsilon)\}. \tag{73}$$

We examine a difference $Diff$ = (34)-(36).

$$\begin{aligned} & Diff \\ = & \int_0^1 \frac{dt}{t}\Pi(Q^2,tQ^2)\int_0^1 dx\frac{\alpha((1-x)tQ^2)}{2\pi}P(x)(x(1-t))^{n-1}\theta(1-x-t)\theta((1-x)t-\epsilon) \\ & \{\exp[\int_0^t \frac{dt'}{t'}\int_0^1 dx'\frac{\alpha((1-x')t'Q^2)}{2\pi}P(x')(x'^{n-1}-1)\theta(1-x'-t')\theta((1-x')t'-\epsilon)] \\ - & \exp[\int_0^t \frac{dt'}{t'}\int_0^1 dx'\frac{\alpha((1-x')t'Q^2)}{2\pi}P(x')((x'(1-t'))^{n-1}-1) \\ & \qquad\qquad \theta(1-x'-t')\theta((1-x')t'-\epsilon)]\}. \end{aligned} \tag{74}$$

If the exponential is expanded by α,

$$\begin{aligned} & Diff \\ \approx & \int_0^1 \frac{dt}{t}\int_0^1 dx\frac{\alpha((1-x)tQ^2)}{2\pi}P(x)(x(1-t))^{n-1}\theta(1-x-t)\theta((1-x)t-\epsilon) \\ & [\int_0^t \frac{dt'}{t'}\int_0^1 dx'\frac{\alpha((1-x')t'Q^2)}{2\pi}P(x')x'^{n-1}(1-(1-t')^{n-1})) \\ & \qquad\qquad \theta(1-x'-t')\theta((1-x')t'-\epsilon)]. \end{aligned} \tag{75}$$

Here we would like to show that this difference is not proportional to $\log(\epsilon)$ and finite as n increases. But it is difficult to obtain the analytic expression for Eq.(75) so that we calculate the simpler expression by replacing the running coupling by α_0. This approximation could not change the essential property of Eq.(75). If this expression is finite as ϵ becomes zero, it is not proportional to $\log(\epsilon)$. So we set ϵ zero in Eq.(75). Then the difference is approximated by

$$\begin{aligned} Diff \approx (\frac{\alpha_0}{2\pi})^2 \int_0^1 \frac{dt}{t}\int_0^{1-t} dxP(x)(x(1-t))^{n-1} \\ \times \int_0^t \frac{dt'}{t'}\int_0^{1-t} dx'P(x')x'^{n-1}(1-(1-t')^{n-1}). \end{aligned} \tag{76}$$

This integral is possible to be expressed analytically, which implies that Eq.(75) is finite as ϵ goes to zero. But this expression is too long to understand the property. In order to confirm that Eq.(76) is finite as n increases, we present numerical values

of them for various n in Table 8, which supports our conclusions. Summarizing this appendix, the error by the approximation on $D_s(n,Q^2)$ is of order of α^2 and n^0 so that we can neglect this difference safely.

Appendix C

In this appendix we present the shower algorithm to generate x according to that the moment is given by Eq.(64), which is

$$exp[\int_0^1 dx(x^{n-1}-1)I^f(x)], \quad I^f(x) = -\frac{\alpha_0}{2\pi}\Delta P^A(x).$$

We introduce $\overline{D}^f(n, x_{max})$, which is

$$\overline{D}^f(n, x_{max}) = exp[\int_0^{x_{max}} dx x^{n-1} I^f(x)]. \tag{77}$$

Since the exponential is expanded to be the infinite series,

$$\begin{aligned}\overline{D}^f(n, x_{max}) &= \exp(\int_0^{x_{max}} dx x^{n-1} I^f(x)) = \sum_{k=0}^{\infty}[\int_0^{x_{max}} dx x^{n-1} I^f(x)]^k/k! \\ = \quad \sum_{k=0}^{\infty}\int_0^{x_{max}} & dx_1 x_1^{n-1} I^f(x_1)\int_0^{x_1} dx_2 x_2^{n-1} I^f(x_2)\cdot\cdot\int_0^{x_{k-1}} dx_k x_k^{n-1} I(x_k),\end{aligned} \tag{78}$$

we can separate contributions of the no-branching and the branching with one or more particles.

$$\overline{D}^f(n, x_{max}) = 1 + \int_0^{x_{max}} dx x^{n-1} I^f(x)\overline{D}^f(n, x). \tag{79}$$

By this equation the probability for no-branching $Pr_{\mathrm{N}}(x_{max})$ is that

$$Pr_{\mathrm{N}}(x_{max}) = 1/\overline{D}^f(1, x_{max}) = \exp(-\int_0^{x_{max}} dx I^f(x)). \tag{80}$$

While the probability $Pr_{\mathrm{E}}(x)dx$ for the first branching at $[x, x+dx](x < x_{max})$ is given by

$$\begin{aligned}Pr_{\mathrm{E}}(x)dx &= dx I^f(x)\overline{D}^f(1, x)/\overline{D}^f(1, x_{max}) \\ &= dx I^f(x)\exp(-\int_x^{x_{max}} dy I^f(y)).\end{aligned} \tag{81}$$

By replacing x_{max} by x, we can repeat the use of Eqs.(80) and (81). Then we obtain the following iterative algorithm for the shower.

- step(1) Set $x_m = 1$ and $x_b = 1$.
- step(2) Calculate a probability of stop, $Pr_N(x_m) = \exp(-\int_0^{x_m} dy I^f(y))$.
- step(3) Generate a uniform random number $\xi (0 < \xi < 1)$. If ξ is less than the probability, go to step(5).
- step(4) Calculate x that satisfies

  $$\xi = \exp(-\int_0^x dy I^f(y)).$$

  Replace x_b by $x_b x$. Then set $x_m = x$ and go back to step(2).
- step(5) Finish generating one event. Then calculate x_b^{n-1} for various n and accumulate them.

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	$Q^2 = 10^4 GeV^2$		$Q^2 = 10^6 GeV^2$	
mom	Analytic	Showers	Analytic	Showers
2	0.96536	0.96539 ±0.16E-05	0.95836	0.95833 ±0.86E-06
3	0.94740	0.94745 ±0.20E-05	0.93670	0.93669 ±0.11E-05
4	0.93525	0.93531 ±0.21E-05	0.92200	0.92201 ±0.13E-05
5	0.92609	0.92617 ±0.22E-05	0.91091	0.91093 ±0.14E-05
6	0.91878	0.91886 ±0.23E-05	0.90202	0.90205 ±0.16E-05
7	0.91270	0.91279 ±0.24E-05	0.89464	0.89466 ±0.17E-05
8	0.90751	0.90761 ±0.25E-05	0.88833	0.88836 ±0.18E-05
9	0.90301	0.90311 ±0.26E-05	0.88283	0.88287 ±0.19E-05
10	0.89902	0.89913 ±0.27E-05	0.87797	0.87801 ±0.20E-05
11	0.89546	0.89558 ±0.28E-05	0.87363	0.87366 ±0.20E-05
12	0.89225	0.89237 ±0.29E-05	0.86969	0.86973 ±0.21E-05
13	0.88932	0.88944 ±0.29E-05	0.86611	0.86615 ±0.22E-05
14	0.88663	0.88676 ±0.30E-05	0.86282	0.86286 ±0.22E-05
15	0.88415	0.88429 ±0.31E-05	0.85978	0.85982 ±0.23E-05
16	0.88186	0.88199 ±0.31E-05	0.85696	0.85700 ±0.23E-05
17	0.87971	0.87985 ±0.32E-05	0.85433	0.85437 ±0.24E-05
18	0.87771	0.87785 ±0.32E-05	0.85187	0.85191 ±0.24E-05
19	0.87583	0.87597 ±0.33E-05	0.84956	0.84959 ±0.24E-05
20	0.87405	0.87420 ±0.33E-05	0.84738	0.84741 ±0.25E-05
30	0.86047	0.86065 ±0.35E-05	0.83061	0.83063 ±0.27E-05
40	0.85130	0.85150 ±0.35E-05	0.81923	0.81925 ±0.29E-05
50	0.84445	0.84467 ±0.35E-05	0.81070	0.81071 ±0.30E-05
60	0.83904	0.83926 ±0.34E-05	0.80392	0.80392 ±0.31E-05
70	0.83458	0.83481 ±0.33E-05	0.79832	0.79832 ±0.32E-05
80	0.83080	0.83105 ±0.32E-05	0.79357	0.79356 ±0.33E-05
90	0.82754	0.82780 ±0.32E-05	0.78945	0.78944 ±0.34E-05
100	0.82468	0.82494 ±0.31E-05	0.78583	0.78582 ±0.34E-05

Table 1: Numerical results of analytic calculations and the shower for the effective LL order P-function. Analytic results are given by Eq.(38). Here $\mu^2 = 0.25 \times 10^{-6}$GeV, and $\alpha_0 = 1/137$. The same values for them are used in coming tables.

mom	$x_{b1}(1-t_1)x_{b2}(1-t_2)$	$(x_{b1}-t_1)(x_{b2}-t_2)$	q^2/S
2	0.93212 ±0.34E-04	0.93118 ±0.34E-04	0.93112 ±0.34E-04
3	0.89787 ±0.48E-04	0.89725 ±0.48E-04	0.89721 ±0.48E-04
4	0.87504 ±0.58E-04	0.87458 ±0.58E-04	0.87455 ±0.58E-04
5	0.85804 ±0.66E-04	0.85767 ±0.66E-04	0.85765 ±0.66E-04
6	0.84457 ±0.71E-04	0.84426 ±0.71E-04	0.84424 ±0.71E-04
7	0.83346 ±0.75E-04	0.83320 ±0.75E-04	0.83318 ±0.75E-04
8	0.82405 ±0.79E-04	0.82382 ±0.79E-04	0.82381 ±0.79E-04
9	0.81590 ±0.81E-04	0.81570 ±0.81E-04	0.81569 ±0.81E-04
10	0.80874 ±0.84E-04	0.80856 ±0.84E-04	0.80855 ±0.84E-04
11	0.80236 ±0.85E-04	0.80220 ±0.86E-04	0.80219 ±0.86E-04
12	0.79663 ±0.87E-04	0.79648 ±0.87E-04	0.79647 ±0.87E-04
13	0.79142 ±0.88E-04	0.79128 ±0.88E-04	0.79127 ±0.88E-04
14	0.78666 ±0.90E-04	0.78653 ±0.90E-04	0.78652 ±0.90E-04
15	0.78228 ±0.91E-04	0.78216 ±0.91E-04	0.78215 ±0.91E-04
16	0.77822 ±0.92E-04	0.77811 ±0.92E-04	0.77811 ±0.92E-04
17	0.77446 ±0.92E-04	0.77435 ±0.92E-04	0.77435 ±0.92E-04
18	0.77094 ±0.93E-04	0.77084 ±0.93E-04	0.77084 ±0.93E-04
19	0.76765 ±0.94E-04	0.76755 ±0.94E-04	0.76755 ±0.94E-04
20	0.76455 ±0.95E-04	0.76446 ±0.95E-04	0.76446 ±0.95E-04
30	0.74103 ±0.99E-04	0.74097 ±0.99E-04	0.74096 ±0.99E-04
40	0.72535 ±0.10E-03	0.72531 ±0.10E-03	0.72531 ±0.10E-03
50	0.71376 ±0.11E-03	0.71373 ±0.11E-03	0.71373 ±0.11E-03
60	0.70466 ±0.11E-03	0.70463 ±0.11E-03	0.70462 ±0.11E-03
70	0.69720 ±0.11E-03	0.69718 ±0.11E-03	0.69718 ±0.11E-03
80	0.69092 ±0.11E-03	0.69090 ±0.11E-03	0.69090 ±0.11E-03
90	0.68553 ±0.11E-03	0.68551 ±0.11E-03	0.68551 ±0.11E-03
100	0.68080 ±0.11E-03	0.68079 ±0.11E-03	0.68078 ±0.11E-03

Table 2: Numerical results of the shower and the generator in the effective LL order. Columns with $x_{b1}(1-t_1)x_{b2}(1-t_2)$ and $(x_{b1}-t_1)(x_{b2}-t_2)$ are results of the shower for these variables, while the third columns are results of the generator for q^2/S.

mom	$D_d(n,Q^2)/(\alpha_0/2\pi)$	$D_d(n,Q^2)$
1	1.6882	0.1961E-02
2	0.8009	0.9305E-03
3	0.5477	0.6363E-03
4	0.4223	0.4906E-03
5	0.3455	0.4014E-03
6	0.2931	0.3405E-03
7	0.2548	0.2960E-03
8	0.2255	0.2620E-03
9	0.2024	0.2351E-03
10	0.1836	0.2132E-03
11	0.1680	0.1952E-03
12	0.1549	0.1799E-03
13	0.1437	0.1669E-03
14	0.1340	0.1556E-03
15	0.1255	0.1458E-03
16	0.1181	0.1372E-03
17	0.1115	0.1295E-03
18	0.1056	0.1226E-03
19	0.1003	0.1165E-03
20	0.0955	0.1109E-03
30	0.0646	0.7501E-04
40	0.0488	0.5670E-04
50	0.0392	0.4556E-04
60	0.0328	0.3809E-04
70	0.0282	0.3272E-04
80	0.0247	0.2869E-04
90	0.0220	0.2553E-04
100	0.0198	0.2300E-04

Table 3: Numerical results on $D_d(n,Q^2)$ of Eq.(57) in Sect.8. They are shown in the third column, while values in the second one are calculated by dividing them by $\alpha_0/2\pi$. While in other tables the moment starts from 2, results for moment 1 are included here.

mom	$D^f(n)^{-1}$	$D^{f,A}(n)$	$D^{f,A}(n)_{shower}$
2	0.99714	0.99701	0.99700±0.82E-05
3	0.99450	0.99443	0.99442±0.15E-04
4	0.99221	0.99217	0.99216±0.20E-04
5	0.99017	0.99015	0.99015±0.24E-04
6	0.98835	0.98834	0.98833±0.27E-04
7	0.98669	0.98668	0.98667±0.30E-04
8	0.98516	0.98516	0.98515±0.32E-04
9	0.98375	0.98375	0.98374±0.35E-04
10	0.98243	0.98244	0.98243±0.37E-04
11	0.98120	0.98120	0.98119±0.38E-04
12	0.98004	0.98004	0.98003±0.40E-04
13	0.97894	0.97894	0.97893±0.42E-04
14	0.97790	0.97790	0.97789±0.43E-04
15	0.97690	0.97691	0.97690±0.44E-04
16	0.97596	0.97596	0.97595±0.45E-04
17	0.97505	0.97506	0.97505±0.46E-04
18	0.97418	0.97419	0.97418±0.48E-04
19	0.97334	0.97335	0.97334±0.49E-04
20	0.97254	0.97254	0.97253±0.49E-04
30	0.96577	0.96578	0.96577±0.57E-04
40	0.96054	0.96055	0.96054±0.62E-04
50	0.95624	0.95626	0.95625±0.66E-04
60	0.95258	0.95260	0.95258±0.68E-04
70	0.94938	0.94941	0.94939±0.71E-04
80	0.94653	0.94656	0.94654±0.73E-04
90	0.94395	0.94399	0.94397±0.75E-04
100	0.94161	0.94165	0.94162±0.76E-04

Table 4: Numerical results on the compensation for the Q^2-independent contributions. The second figures show $D^f(n)^{-1}$, while the third ones are approximated contributions by $D^{f,A}(n)$. Results by the shower algorithm are given in the last column

mom	LL order(analyt.)	NLL order(analyt.)	$x_{b1}(1-t_1)x_{b2}(1-t_2)$	q^2/S
2	0.92649	0.92677	0.92655±0.37E-04	0.92554±0.38E-04
3	0.88754	0.88799	0.88789±0.51E-04	0.88723±0.51E-04
4	0.86085	0.86143	0.86140±0.57E-04	0.86090±0.57E-04
5	0.84055	0.84124	0.84124±0.61E-04	0.84085±0.61E-04
6	0.82421	0.82498	0.82500±0.63E-04	0.82467±0.63E-04
7	0.81056	0.81141	0.81143±0.65E-04	0.81115±0.65E-04
8	0.79886	0.79977	0.79979±0.67E-04	0.79955±0.67E-04
9	0.78863	0.78960	0.78963±0.68E-04	0.78941±0.68E-04
10	0.77956	0.78058	0.78060±0.69E-04	0.78041±0.69E-04
11	0.77142	0.77249	0.77251±0.70E-04	0.77233±0.70E-04
12	0.76404	0.76516	0.76517±0.71E-04	0.76501±0.71E-04
13	0.75730	0.75846	0.75847±0.72E-04	0.75832±0.72E-04
14	0.75110	0.75229	0.75230±0.73E-04	0.75217±0.73E-04
15	0.74536	0.74659	0.74660±0.74E-04	0.74647±0.74E-04
16	0.74002	0.74128	0.74129±0.74E-04	0.74117±0.75E-04
17	0.73504	0.73633	0.73633±0.75E-04	0.73622±0.75E-04
18	0.73036	0.73168	0.73168±0.76E-04	0.73158±0.76E-04
19	0.72596	0.72731	0.72730±0.77E-04	0.72721±0.77E-04
20	0.72180	0.72318	0.72317±0.77E-04	0.72308±0.77E-04
30	0.68965	0.69123	0.69120±0.81E-04	0.69114±0.81E-04
40	0.66759	0.66932	0.66928±0.83E-04	0.66924±0.83E-04
50	0.65093	0.65277	0.65271±0.83E-04	0.65268±0.83E-04
60	0.63759	0.63952	0.63946±0.84E-04	0.63943±0.84E-04
70	0.62652	0.62853	0.62846±0.84E-04	0.62843±0.84E-04
80	0.61707	0.61914	0.61907±0.85E-04	0.61905±0.85E-04
90	0.60885	0.61098	0.61090±0.85E-04	0.61089±0.85E-04
100	0.60158	0.60377	0.60369±0.86E-04	0.60367±0.86E-04

Table 5: Numerical results in the NLL order. The second figures show moments in the LL order, while the third ones are the NLL order results. Both are calculated analytically. Results on $x_{b1}(1-t_1)x_{b2}(1-t_2)$ and q^2/S obtained by our generator are given in the last two columns. $S = 10^4\text{GeV}^2$, $\mu^2 = 0.25 \times 10^{-6}\text{GeV}^2$.

mom	LL order(analyt.)	NLL order(analyt.)	$x_{b1}(1-t_1)x_{b2}(1-t_2)$	q^2/S
2	0.91309	0.91341	0.91318 ±0.53E-04	0.91216 ±0.55E-04
3	0.86757	0.86809	0.86795 ±0.66E-04	0.86729 ±0.67E-04
4	0.83658	0.83726	0.83717 ±0.74E-04	0.83668 ±0.74E-04
5	0.81315	0.81394	0.81387 ±0.79E-04	0.81348 ±0.80E-04
6	0.79436	0.79525	0.79518 ±0.84E-04	0.79486 ±0.84E-04
7	0.77872	0.77969	0.77962 ±0.87E-04	0.77934 ±0.87E-04
8	0.76535	0.76639	0.76632 ±0.90E-04	0.76608 ±0.90E-04
9	0.75370	0.75481	0.75472 ±0.93E-04	0.75451 ±0.93E-04
10	0.74338	0.74455	0.74446 ±0.95E-04	0.74427 ±0.95E-04
11	0.73415	0.73537	0.73527 ±0.97E-04	0.73510 ±0.97E-04
12	0.72579	0.72706	0.72696 ±0.98E-04	0.72681 ±0.98E-04
13	0.71818	0.71949	0.71938 ±0.10E-03	0.71924 ±0.10E-03
14	0.71118	0.71253	0.71242 ±0.10E-03	0.71229 ±0.10E-03
15	0.70471	0.70610	0.70598 ±0.10E-03	0.70586 ±0.10E-03
16	0.69871	0.70013	0.70001 ±0.10E-03	0.69990 ±0.10E-03
17	0.69311	0.69456	0.69444 ±0.10E-03	0.69433 ±0.10E-03
18	0.68786	0.68935	0.68922 ±0.11E-03	0.68912 ±0.11E-03
19	0.68293	0.68444	0.68431 ±0.11E-03	0.68422 ±0.11E-03
20	0.67827	0.67982	0.67968 ±0.11E-03	0.67959 ±0.11E-03
30	0.64245	0.64421	0.64405 ±0.11E-03	0.64399 ±0.11E-03
40	0.61806	0.61997	0.61979 ±0.11E-03	0.61975 ±0.11E-03
50	0.59973	0.60175	0.60156 ±0.11E-03	0.60153 ±0.11E-03
60	0.58513	0.58724	0.58705 ±0.11E-03	0.58702 ±0.11E-03
70	0.57304	0.57524	0.57503 ±0.11E-03	0.57501 ±0.11E-03
80	0.56277	0.56503	0.56482 ±0.11E-03	0.56480 ±0.11E-03
90	0.55385	0.55616	0.55595 ±0.11E-03	0.55594 ±0.11E-03
100	0.54599	0.54835	0.54814 ±0.11E-03	0.54812 ±0.11E-03

Table 6: The same as Table 5 except S. Here $S = 10^6\mathrm{GeV}^2$.

	$S = 10^4\mathrm{GeV}^2$		$S = 10^6\mathrm{GeV}^2$		$S = 10^8\mathrm{GeV}^2$	
mom	generator	shower	generator	shower	generator	shower
2	-0.13218%	-0.02363%	-0.13773%	-0.02573%	-0.13908%	-0.02333%
3	-0.08570%	-0.01171%	-0.09262%	-0.01624%	-0.09109%	-0.01249%
4	-0.06095%	-0.00395%	-0.06951%	-0.01075%	-0.06612%	-0.00565%
5	-0.04660%	-0.00012%	-0.05688%	-0.00872%	-0.05194%	-0.00254%
6	-0.03770%	0.00182%	-0.04917%	-0.00843%	-0.04318%	-0.00130%
7	-0.03167%	0.00271%	-0.04438%	-0.00898%	-0.03751%	-0.00107%
8	-0.02738%	0.00300%	-0.04110%	-0.00979%	-0.03350%	-0.00136%
9	-0.02432%	0.00291%	-0.03882%	-0.01073%	-0.03063%	-0.00180%
10	-0.02203%	0.00269%	-0.03720%	-0.01182%	-0.02859%	-0.00239%
11	-0.02032%	0.00233%	-0.03617%	-0.01278%	-0.02700%	-0.00300%
12	-0.01895%	0.00196%	-0.03521%	-0.01375%	-0.02562%	-0.00347%
13	-0.01767%	0.00158%	-0.03461%	-0.01473%	-0.02447%	-0.00396%
14	-0.01675%	0.00133%	-0.03410%	-0.01558%	-0.02386%	-0.00474%
15	-0.01607%	0.00094%	-0.03385%	-0.01643%	-0.02307%	-0.00524%
16	-0.01538%	0.00054%	-0.03342%	-0.01714%	-0.02269%	-0.00590%
17	-0.01467%	0.00027%	-0.03326%	-0.01785%	-0.02229%	-0.00641%
18	-0.01435%	-0.00014%	-0.03307%	-0.01857%	-0.02187%	-0.00678%
19	-0.01389%	-0.00041%	-0.03302%	-0.01914%	-0.02159%	-0.00730%
20	-0.01369%	-0.00083%	-0.03310%	-0.01986%	-0.02160%	-0.00798%
30	-0.01288%	-0.00420%	-0.03415%	-0.02530%	-0.02216%	-0.01300%
40	-0.01300%	-0.00642%	-0.03549%	-0.02871%	-0.02369%	-0.01672%
50	-0.01348%	-0.00827%	-0.03673%	-0.03141%	-0.02525%	-0.01966%
60	-0.01407%	-0.00969%	-0.03814%	-0.03355%	-0.02690%	-0.02226%
70	-0.01448%	-0.01082%	-0.03911%	-0.03529%	-0.02850%	-0.02451%
80	-0.01486%	-0.01163%	-0.04018%	-0.03681%	-0.02968%	-0.02619%
90	-0.01522%	-0.01228%	-0.04100%	-0.03794%	-0.03082%	-0.02766%
100	-0.01557%	-0.01292%	-0.04194%	-0.03921%	-0.03194%	-0.02892%

Table 7: Differences between the analytic results and the generator's or the shower's ones. The ratios of them are shown. The columns with q^2/S denote the ratios by the generator. We neglect the statistical error. Here the shower and the generator compensate for the Q^2-independent contributions.

mom	Eq.(76)/$(\alpha_0/2\pi)^2$	Eq.(76)
1	3.3366	0.4503E-05
2	4.4792	0.6045E-05
3	5.0860	0.6864E-05
4	5.4672	0.7378E-05
5	5.7301	0.7733E-05
6	5.9227	0.7993E-05
7	6.0702	0.8192E-05
8	6.1868	0.8350E-05
9	6.2813	0.8477E-05
10	6.3594	0.8583E-05
11	6.4252	0.8671E-05
12	6.4812	0.8747E-05
13	6.5296	0.8812E-05
14	6.5718	0.8869E-05
15	6.6089	0.8919E-05
16	6.6418	0.8964E-05
17	6.6712	0.9003E-05
18	6.6976	0.9039E-05
19	6.7214	0.9071E-05
20	6.7430	0.9100E-05
30	6.8840	0.9291E-05
40	6.9574	0.9390E-05
50	7.0023	0.9450E-05
60	7.0327	0.9491E-05
70	7.0546	0.9521E-05
80	7.0711	0.9543E-05
90	7.0840	0.9560E-05
100	7.0944	0.9574E-05

Table 8: Numerical results on $Diff$ of Eq.(76) in Appendix C. These differences are shown in the third column, while values in the second one are calculated by dividing them by $(\alpha_0/2\pi)^2$.

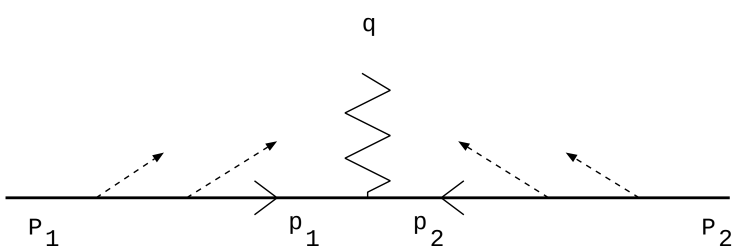

Fig.1

Figure 1: The schematics of the annihilation process. The electron(positron) has the momentum $P_1(P_2)$ initially, and does $p_1(p_2)$ after the branching process. The momentum of the virtual photon is denoted by q.